JOURNAL OF ALGEBRA 38, 280-308 (1976)

Representations of the Symmetric Groups over the Field of Order 2

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Communicated by Walter Feit

Received May 29, 1974

INTRODUCTION

In his book [3], Robinson gave a method for determining all the p-modular irreducible characters of the symmetric groups for odd primes p. However, his results for the prime 2 were incorrect. In [4], he gave an explanation of why his methods were wrong and gave amended 2-modular decomposition matrices for S_n up to $n = 8$ and a decomposition matrix for S_9 which he believed to be correct. His method is so laborious that it is, in practice, impossible to push the results much further. In this paper, the problem is approached from an entirely different point of view. The parts of the decomposition matrices corresponding to characters of types $[n - m, m]$ and $[n - m - 1, m, 1]$ are found for all symmetric groups.

Although the complications will multiply when the methods used here are extended to characters of other types, undoubtedly they can be used to determine all the 2-modular representations of any particular symmetric group. In the cases we consider, the 2-modular irreducible character turning up from an ordinary character χ is found to be the modular character of a module, whose 2-modular character is χ, modulo the kernel of a natural bilinear form. Another feature common to both types considered is that the parts of the decomposition matrices involved are very similar.

More modular characters of S_n can be found from the results given here, by using the knowledge of modular irreducibles of S_{n-1} and the methods used in [1]. This is illustrated in the final section, where Robinson's results for S_8 and S_9 are checked and all the modular irreducibles of S_{10} are found.

1. Preliminary and Quoted Rfsults

Throughout this paper "modular" will mean "2-modular."

If ϕ and χ are modular characters, we shall write $\varphi \subseteq \chi$ if φ is a constituent of χ, and if M is a module, we write $\varphi \subseteq M$ if φ is a constituent of the character of M.

Definition. We say that a set of modular characters $\chi_0, \chi_1, \ldots, \chi_{k-1}$ is *ascending* if for each i there is a modular irreducible character $\varphi(\chi_i)$ that is a constituent of χ_i with multiplicity 1, but not a constituent of any χ_j for $j < i$ and the only constituents of χ_i are some of $\varphi(\chi_0), \varphi(\chi_1), \ldots, \varphi(\chi_i)$.

It is proved in [3], that if $\chi_i = [n-i, i]$ for $0 \leqslant i < n/2$ and $\chi_j{}^* = [n-j-1, j, 1]$ for $1 < j < (n-1)/2$, then the set $\chi_0, \chi_1, \ldots, \chi_2{}^*, \chi_3{}^*, \ldots$ is ascending. We define $\varphi[n-m, m]$ and $\varphi[n-m-1, m, 1]$ as in the above definition of ascending and let $\varphi[n-m, m] = 0$ if $m \geqslant n/2$ and $\varphi[n-m-1, m, 1] = 0$ if $m \geqslant (n-1)/2$, or if $m \leqslant 1$.

The set of characters whose Young diagrams have rows of different lengths, form a spanning set for the space of 2-modular characters. Thus, for small n, all or most of the 2-modular irreducibles are given by $\varphi[n-m, m]$ and $\varphi[n-m-1, m, 1]$, as m varies.

Let $[n-m, m] = 0$ if $m > n/2$ or $m < 0$, and let $[n-m-1, m, 1] = 0$ if $m > (n-1)/2$ or $m < 1$. This convention is especially useful when restricting or inducing characters from one symmetric group to the next. If χ is a character of S_n, the notation $\chi\!\downarrow$ is used for the restriction of χ to S_{n-1} and $\chi\!\uparrow$ for the character induced up to S_{n+1}. Then, for instance, $[n-m, m]\!\downarrow = [n-m-1, m] + [n-m, m-1]$, even when $m = 0$ or $n/2$.

It is wellknown that two ordinary irreducible characters of S_n belong to the same 2-block if and only if they have the same 2-core. Hence, for instance, if n is odd, $[n-i, i]$ and $[n-j, j]$ belong to the same 2-block if and only if $i + j$ is even.

Since it is the trivial modular character whose multiplicity in an ordinary character is most difficult to determine, the next theorem is of interest.

Theorem 1.1. *Suppose G is any finite group and φ is a 2-modular irreducible character of G. Then, either φ is the trivial character, or φ is not real, or the degree of φ is even.*

Proof. Suppose that k is a field of characteristic 2 and G is a finite absolutely irreducible subgroup of $GL(n, k)$ with $n > 1$. Suppose, too, that for every x in G, $\mathrm{tr}(x) = \mathrm{tr}(x'^{-1})$, where x' means the transpose of x. We must show that n is then even.

There is a nonsingular matrix a such that for all x in G, $x = ax'^{-1}a^{-1}$. By Schur's lemma, $a' = \mu a$, where μ is an element of k. Therefore, $a = a'' = (\mu a') = \mu a' = \mu^2 a$, so $\mu = 1$ and $a = a'$. For $\mathbf{v}$ and $\mathbf{w}$ elements of k^n, let $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}a\mathbf{w}'$. By construction, this is a G-invariant symmetric bilinear form. If $V = \{\mathbf{v} \mid \langle \mathbf{v}, \mathbf{v} \rangle = 0\}$, then V is an invariant subspace of k^n of codimension at most 1. Therefore, $V = k^n$ and $\langle\ ,\ \rangle$ is alternating. Hence, n is even. ∎

COROLLARY 1.2. *All the 2-modular irreducible characters of S_n, except the trivial one, have even degree. If χ is a modular character of S_n, the trivial character is a constituent of χ with even multiplicity if* $\deg \chi$ *is even, or with odd multiplicity if* $\deg \chi$ *is odd.*

2. THE MODULES V_n^m AND E_n^m

Throughout this section, the reader might like to refer to the example given after Lemma 2.7.

Let n be fixed in this section, and assume $m \leqslant n/2$.

DEFINITIONS. (i) An unordered m-tuple $(i_1, \ldots, i_m)$ such that all the i_j's are different and belong to $\{1, \ldots, n\}$ is called an *m-tuple.*

(ii) For $0 \leqslant m \leqslant n/2$, let V_n^m be the vector space over the field of 2 elements whose basis elements are the $\binom{n}{m}$ different m-tuples. Note that V_n^0 has just two elements, $\mathbf{0}$ and the 0-tuple ( ).

A general element of V_n^m is of the form $\mathbf{v} = \sum \mathbf{w}_r$, where the $\mathbf{w}_r$'s are different m-tuples. We call the $\mathbf{w}_r$'s that are involved in $\mathbf{v}$ the *words* of $\mathbf{v}$. We write $\mathbf{w} \in \mathbf{v}$ if $\mathbf{w}$ is a word of $\mathbf{v}$. The action of an element σ of S_n on this general element $\mathbf{v}$ of V_n^m is defined in the obvious way, namely $\sigma(\mathbf{v}) = \sum \sigma(\mathbf{w}_r)$; if $\mathbf{w}_r = (i_1, \ldots, i_m)$, then $\sigma(\mathbf{w}_r) = (\sigma(i_1), \ldots, \sigma(i_m))$ and if $\mathbf{w}_r = (\ )$ then $\sigma(\mathbf{w}_r) = (\ )$. This turns V_n^m into an S_n module.

Suppose $\mathbf{w}$ is a j-tuple and $\mathbf{u}$ is an m-tuple. We say $i \in \mathbf{w}$ if $\mathbf{w} = (i, i_2, \ldots, i_j)$ for some $i_2, \ldots, i_j$. Then $\mathbf{u} \cap \mathbf{w}$ is defined as $(i_1, \ldots, i_k)$, where $\{i_1, \ldots, i_k\} = \{i \mid i \in \mathbf{u} \text{ and } i \in \mathbf{w}\}$. If $\mathbf{u} \cap \mathbf{w} = \mathbf{u}$, we say $\mathbf{u} \subseteq \mathbf{w}$. If $\mathbf{u} \cap \mathbf{w} = (\ )$, we say $\mathbf{u}$ *misses* $\mathbf{w}$, and write $(\mathbf{u}, \mathbf{w})$ for $(i_1, \ldots, i_k)$, where $\{i_1, \ldots, i_k\} = \{i \mid i \in \mathbf{u} \text{ or } i \in \mathbf{w}\}$.

If $\mathbf{v}$ is an element of V_n^m and $i \in \mathbf{w}$ for some $\mathbf{w} \in \mathbf{v}$, we say *$i$ is involved in* $\mathbf{v}$.

The natural symmetric bilinear form $\langle\ ,\ \rangle$ is put on V_n^m. That is, for m-tuples $\mathbf{w}_1$ and $\mathbf{w}_2$ (basis elements of V_n^m), we let $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 1$ if $\mathbf{w}_1 = \mathbf{w}_2$ and 0 otherwise; we extend this to be a bilinear form for general elements of V_n^m.

DEFINITIONS. Let $\mathbf{w}$ be a fixed k-tuple with $k \geqslant 0$.

(i) For $m \geqslant 0$, $\mathbf{g}_n{}^m\{\mathbf{w}\}$ is defined by

$$\begin{aligned} \mathbf{g}_n{}^m\{\mathbf{w}\} &= \sum \{\mathbf{u} \mid \mathbf{u} \text{ is an } m\text{-tuple} \subseteq \mathbf{w}\} \qquad \text{if} \qquad m \leqslant k \\ &= \sum \{\mathbf{u} \mid \mathbf{u} \text{ is an } m\text{-tuple} \supseteq \mathbf{w}\} \qquad \text{if} \qquad m \geqslant k. \end{aligned}$$

(ii) For $m \geqslant 0$, $\mathbf{g}_n^{-m}\{\mathbf{w}\}$ is defined by

$$\mathbf{g}_n^{-m}\{\mathbf{w}\} = \sum \{\mathbf{u} \mid \mathbf{u} \text{ is an } m\text{-tuple missing } \mathbf{w}\}$$

we shall also write this as $\mathbf{g}_n{}^m\{-\mathbf{w}\}$. Note that $\mathbf{g}_n{}^0\{\mathbf{w}\} = (\ )$.

Geometrically, we may regard $\mathbf{g}_n{}^m\{-\mathbf{w}\}$, for instance, as "the complete graph of m-tuples missing $\mathbf{w}$."

DEFINITION. $G_n{}^m\{k\}$ is defined for $k \geqslant 0$ to be the submodule of $V_n^{|m|}$, spanned by elements of the form $\mathbf{g}_n{}^m\{\mathbf{w}\}$ as $\mathbf{w}$ ranges over all k-tuples. For $k \leqslant 0$, we define $G_n{}^m\{k\} = G_n^{-m}\{-k\}$.

DEFINITION. $\theta_k{}^m$ is defined only for $m \geqslant 0$, but for any k and $\theta_k{}^m$ is the module homomorphism from $V_n{}^m$ to $V_n^{|k|}$ defined by taking a basic m-tuple $\mathbf{w}$ of $V_n{}^m$ to $\mathbf{g}_n{}^k\{\mathbf{w}\}$.

In the case $0 \leqslant k \leqslant m$, for example, an element $\mathbf{v}$ of $V_n{}^m$ belongs to Ker $\theta_k{}^m$ if and only if each k-tuple is contained in an even number of words of $\mathbf{v}$.

THEOREM 2.1.

(i) $\operatorname{Im} \theta_k{}^m = G_n{}^k\{m\}$, *for* $m \geqslant 0$

(ii) $\operatorname{Ker} \theta_k{}^m = (G_n{}^m\{k\})^\perp$ *for* $m \geqslant 0$

(iii) $G_n{}^k\{m\} \cong G_n{}^m\{k\}$.

Proof. The first result is obvious from the definition. Now suppose $0 \leqslant k \leqslant m$. Then, $\mathbf{u}_1 + \cdots + \mathbf{u}_s \in \operatorname{Ker} \theta_k{}^m \Leftrightarrow$ each k-tuple is contained in an even number of $\mathbf{u}_1, \dots, \mathbf{u}_s \Leftrightarrow$ for all k-tuples $\mathbf{w}$, $\mathbf{u}_1 + \cdots + \mathbf{u}_s \in (\mathbf{g}_n{}^m\{\mathbf{w}\})^\perp \Leftrightarrow$ $\mathbf{u}_1 + \cdots + \mathbf{u}_s \in (G_n{}^m\{k\})^\perp$. Thus, the second result follows in this case. The cases $m \leqslant k$ and $k < 0$ can be proved in a similar way. The third result follows at once if $m \geqslant 0$. But since $G_n{}^m\{k\} = G_n^{-m}\{-k\}$, the conclusion is also true when $m < 0$. ▌

LEMMA 2.2. *Let* $\mathbf{s}$ *be a fixed k-tuple with* $0 \leqslant k < m$ *and let* $\mathbf{s}'$ *be a fixed subset of* $\mathbf{s}$. *Then the number of words* $\mathbf{w}$ *of* $\mathbf{v}$ *such that* $\mathbf{w} \cap \mathbf{s} = \mathbf{s}'$ *is even in each of the following cases:*

(i) $\mathbf{v} \in \bigcap_{i=0}^{k} \operatorname{Ker} \theta_i^m$

(ii) $\mathbf{v} \in \bigcap_{i=0}^{k} \operatorname{Ker} \theta_{-i}^m$.

Proof. Consider case (i). The result is certainly true if $\mathbf{s}' = \mathbf{s}$, since $\mathbf{v} \in \operatorname{Ker} \theta_k^m$. If $\mathbf{s}' < \mathbf{s}$, the number of $\mathbf{w} \in \mathbf{v}$ such that $\mathbf{w} \supseteq \mathbf{s}'$ is even by the definition of $\mathbf{v}$; of these, an even number satisfy $\mathbf{w} \cap \mathbf{s} > \mathbf{s}'$ by induction. Thus, the required result follows. Case (ii) can be proved similarly. ∎

Taking $\mathbf{s}' = (\ )$ in (i) and $\mathbf{s}' = \mathbf{s}$ in (ii), we have the immediate

COROLLARY 2.3. *If* $0 \leqslant k < m$, *then* $\bigcap_{i=0}^{k} \operatorname{Ker} \theta_i^m = \bigcap_{i=0}^{k} \operatorname{Ker} \theta_{-i}^m$.

We now give a special notation for one of these intersections and derive a spanning set for it.

DEFINITION. $E_n^m = \bigcap_{i=0}^{m-1} \operatorname{Ker} \theta_i^m = \bigcap_{i=0}^{m-1} \operatorname{Ker} \theta_{-i}^m$.

Note. We adopt the usual convention for empty intersections, so $\bigcap_{i=0}^{-1} \operatorname{Ker} \theta_i^m = V_n^m$. In particular, $E_n^0 = V_n^0 = \{\mathbf{0}, (\ )\}$.

It is clear that if $\mathbf{v} \in E_n^m$ and i is involved in $\mathbf{v}$, then the link of i (that is, the set of $\mathbf{u}$ such that $(i, \mathbf{u}) \in \mathbf{v}$) belongs to E_n^{m-1}.

DEFINITION. A *cross-polytope* (c.p.) *of m-tuples* is defined inductively as follows:

(i) $(\ )$ is a cross-polytope of 0-tuples.

(ii) If $\sum_{i=1}^{r} \mathbf{w}_i$ is a cross-polytope of m-tuples involving the points $\{1, 2,..., 2m\}$, then $\sum_{i=1}^{r} \{(\mathbf{w}_i , 2m + 1) + (\mathbf{w}_i , 2m + 2)\}$ is a cross-polytope of $(m + 1)$-tuples.

(iii) If $\sum_{i=1}^{r} \mathbf{w}_i$ is a cross-polytope of m-tuples, then so is any image of it under the action of elements of S_n .

A c.p. of m-tuples involves 2^m words and $2m$ points.

It is easy to see that any c.p. of m-tuples belongs to E_n^m. We now prove the important

THEOREM 2.4. *E_n^m is spanned by cross-polytopes of m-tuples.*

Proof. The result is certainly true if $m = 0$. Suppose $m > 0$ and $\mathbf{0} \neq \mathbf{v} \in E_n^m$. Without loss, we may assume $(1, 2,..., m) = \mathbf{w} \in \mathbf{v}$. If $\mathbf{v}$ involves just the points 1 to $m + k$ with $k < m$, let $\mathbf{s} = (m + 1,..., m + k)$ and $\mathbf{s}' = (\ )$. Then $\mathbf{w}$ is the unique word of $\mathbf{v}$ such that $\mathbf{w} \cap \mathbf{s} = \mathbf{s}'$. This contradicts Lemma 2.2(i). Therefore, $\mathbf{v}$ involves at least $2m$ letters.

Now, by induction, the link of 1 is the sum of cross-polytopes of $(m - 1)$-tuples. Let $\mathbf{p} = \mathbf{w}_1 + \cdots + \mathbf{w}_r$ be one of these. Since 1 and $\mathbf{p}$ account for just $2m - 1$ points, there is another point, say j, involved in $\mathbf{v}$. Then $(1, \mathbf{w}_1) + \cdots + (1, \mathbf{w}_r) + (j, \mathbf{w}_1) + \cdots + (j, \mathbf{w}_r)$ is a c.p. of m-tuples. For each c.p. of $(m - 1)$-tuples involved in the link of 1, construct a c.p. of m-tuples in this way and add the sum of all of them to $\mathbf{v}$. This yields a new element of $E_n{}^m$ with 1 removed and no new points introduced. Hence, by induction, $\mathbf{v}$ is a sum of cross-polytopes. ∎

If $\mathbf{p}$ is a c.p. of k-tuples on $\{1, 2,..., 2k\}$, then each word of $\mathbf{p}$ has a unique "opposite" word. That is, if $\mathbf{w} \in \mathbf{p}$, there is a unique $\mathbf{u} \in \mathbf{p}$ such that $\mathbf{u} \cap \mathbf{w} = (\ )$. Suppose that $\mathbf{s} \subseteq \{1, 2,..., 2k\}$. If $|\mathbf{s}| < k$, then $\mathbf{s}$ misses an even number of words of $\mathbf{p}$. If $|\mathbf{s}| > k$, s contains an even number of words of $\mathbf{p}$, since its complement in $\{1, 2,..., 2k\}$ misses an even number. If $|\mathbf{s}| = k$, either $\mathbf{s}$ is a word of $\mathbf{p}$ and misses the opposite word, or $\mathbf{s}$ misses no word of $\mathbf{p}$. Hence, we have

LEMMA 2.5. *Suppose $m \geqslant k \geqslant 0$ and $\sum_{i=1}^{r} \mathbf{w}_i$ is a cross-polytope of k-tuples on $\{1, 2,..., 2k\}$. Then $\sum_{i=1}^{r} \mathbf{g}_n{}^m\{-\mathbf{w}_i\} = \sum_{i=1}^{r} \mathbf{g}_n{}^m\{\mathbf{w}_i\} = \sum_{i=1}^{r}\sum_{\mathbf{u}} (\mathbf{u}, \mathbf{w}_i)$, where $\sum \mathbf{u} = \mathbf{g}_n^{m-k}\{-(1, 2,..., 2k)\}$.*

For the moment, assume $m \geqslant k \geqslant 0$ and $K = \bigcap_{i=0}^{k-1} \operatorname{Ker} \theta_i{}^m = \bigcap_{i=0}^{k-1} \operatorname{Ker} \theta_{-i}^m$. Then, it is easy to check that $\theta_k{}^m(K)$ and $\theta_{-k}^m(K)$ are both contained in $E_n{}^k$. However, if $\sum_{i=1}^{r} \mathbf{w}_i$ is a c.p. of k-tuples and $\mathbf{t} = \sum_{i=1}^{r} (\mathbf{w}_i, 2k + 1,..., k + m)$, then $\mathbf{t} \in K$ and by Lemma 2.5, $\theta_k{}^m(\mathbf{t}) = \theta_{-k}^m(\mathbf{t}) = \sum_{i=1}^{r} \mathbf{w}_i$. In view of Theorems 2.4 and 2.1, we have proved

LEMMA 2.6. *If $m \geqslant k \geqslant 0$, then $E_n{}^k = \theta_k{}^m(\bigcap_{i=0}^{k-1} \operatorname{Ker} \theta_i{}^m) = \theta_{-k}^m(\bigcap_{i=0}^{k-1} \operatorname{Ker} \theta_i{}^m)$, and so $E_n{}^k$ is a submodule of both $G_n{}^k\{m\}$ and $G_n{}^k\{-m\}$.*

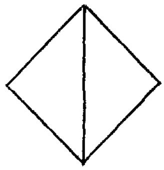
Now we are in a position to generalize Theorem 2.4.

LEMMA 2.7. *Suppose $m \geqslant k \geqslant 0$ and $\mathbf{w}_1 + \cdots + \mathbf{w}_r$ is a cross-polytope of k-tuples on $\{1, 2,..., 2k\}$. Let T_k be the submodule of $V_n{}^m$ generated by $\sum_{i=1}^{r} (\mathbf{w}_i, 2k + 1,..., k + m)$. Then, $T_k = \bigcap_{i=0}^{k-1} \operatorname{Ker} \theta_i{}^m$.*

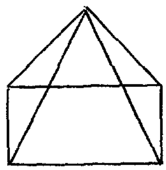
Proof. The result is true for $k = m$ by Theorem 2.4. Assume it is true for $k = j > 0$. Now, we have seen above that $\bigcap_{i=0}^{j-2} \operatorname{Ker} \theta_i{}^m \supseteq T_{j-1}$ and that the images of both of these under θ_{j-1}^m is E_n^{j-1}. But $T_j \subseteq T_{j-1}$, so $T_{j-1} \cap \operatorname{Ker} \theta_{j-1}^m = T_j = \bigcap_{i=0}^{j-1} \operatorname{Ker} \theta_i{}^m$. Therefore, the result is true for $k = j - 1$. ∎

EXAMPLE. Consider the case $n = 6$ and $m = 3$. We may regard $\{1,..., 6\}$ as six points labelled 1,..., 6, and $V_6{}^3$ as the space spanned by the faces of triangles on these points.

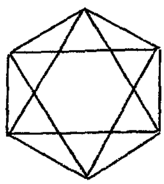
By Lemma 2.7, Ker θ_0^3 is spanned by elements of V_6^3 looking like



Ker $\theta_0^3 \cap$ Ker θ_1^3 is spanned by elements like



$E_6^3 =$ Ker $\theta_0^3 \cap$ Ker $\theta_1^3 \cap$ Ker θ_2^3 is spanned by octahedra:



E_6^3 consists of those elements of V_6^3 that have the following three properties:

(a) The element has an even number of triangles.

(b) Each point is in an even number of triangles.

(c) Each edge is in an even number of triangles.

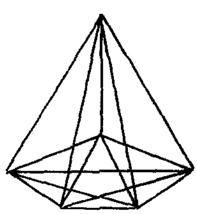
Corollary 2.3 shows that conditions (a), (b), and (c) may be replaced by conditions (a), (b′), and (c′), where (b′) and (c′) are:

(b′) Each point misses an even number of triangles.

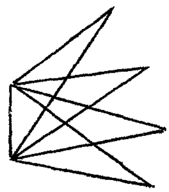
(c′) Each edge misses an even number of triangles.

$G_6^3\{0\}$ is the sum of all the triangles on the six points.

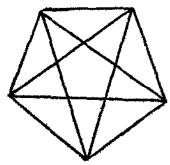
$G_6^3\{1\}$ is spanned by elements like



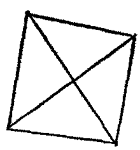
$G_6^3\{2\}$ is spanned by elements like



$G_6^3\{-1\}$ is spanned by elements like

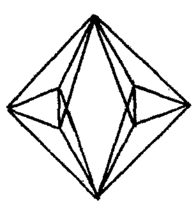


$G_6^3\{-2\}$ is spanned by elements like



By Theorem 2.1(ii), $E_6^{3\perp} = G_6^3\{0\} + G_6^3\{1\} + G_6^3\{2\} = G_6^3\{0\} + G_6^3\{-1\} + G_6^3\{-2\}$.

For an example of an element in E_n^3 that is the sum of more than one octahedron, consider $n = 8$ and



Next, the modular character of E_n^m is found.

THEOREM 2.8. *If* $m \geqslant k \geqslant 0$, *then* $\bigcap_{i=0}^{k-1} \operatorname{Ker} \theta_i^m = [n-k, k] + [n-k-1, k+1] + \cdots + [n-m, m]$. *In particular*, $E_n^m = [n-m, m]$.

Proof. It is straightforward to prove that the permutation character of S_n on $S_m \times S_{n-m}$ is $[n] + [n-1, 1] + \cdots + [n-m, m]$; so this is the character

character of $V_n{}^m$ and the result is true when $k = 0$. Take a counterexample to the theorem with $m \geqslant k > 0$ and with $m + k$ minimal. Then, $\bigcap_{i=0}^{k-2} \operatorname{Ker} \theta_i{}^m = [n-k+1, k-1] + \cdots + [n-m, m]$. But, by Lemma 2.6, $\theta_{k-1}^m(\bigcap_{i=0}^{k-2} \operatorname{Ker} \theta_i{}^m) = \bigcap_{i=0}^{k-2} \operatorname{Ker} \theta_i^{k-1}$, which equals $[n-k+1, k-1]$ by induction. Thus, $\bigcap_{i=0}^{k-1} \operatorname{Ker} \theta_i{}^m = [n-k, k] + \cdots + [n-m, m]$, after all. ∎

We conclude this section with two lemmas that will be useful later.

LEMMA 2.9. *If $-m < j < m$, then $G_n{}^m\{j\}$ is contained in $E_n^{m\perp}$.*

Proof. $E_n^{m\perp} = (\bigcap_{i=0}^{m-1} \operatorname{Ker} \theta_i{}^m)^\perp$, which by Corollary 2.3, equals $(\bigcap_{i=0}^{m-1} \operatorname{Ker} \theta_{-i}^m)^\perp$. Thus, by Theorem 2.1, $E_n^{m\perp} = \sum_{i=0}^{m-1} G_n{}^m\{i\} = \sum_{i=0}^{m-1} G_n{}^m\{-i\}$. ∎

Trivially, $E_n{}^m/E_n{}^m \cap E_n^{m\perp}$ has a submodule of dimension 1 if and only if it has a submodule of codimension 1. The object of the next lemma is to investigate this situation.

LEMMA 2.10.

(i) *If $n/2 > m > 0$, then $E_n{}^m$ does not have a submodule of codimension 1.*

(ii) *$E_n{}^m = E_n{}^m \cap E_n^{m\perp}$ if and only if n is even and $m = n/2$.*

Proof. Assume $n/2 > m > 0$. Let $\sum_{i=1}^r \mathbf{w}_i$ be a c.p. of $(m-1)$-tuples on $\{1, 2,..., 2m-2\}$. Let $\mathbf{p}_1 = \sum_{i=1}^r \{(\mathbf{w}_i, 2m-1) + (\mathbf{w}_i, 2m)\}$ and $\mathbf{p}_2 = \sum_{i=1}^r \{(\mathbf{w}_i, 2m-1) + (\mathbf{w}_i, 2m+1)\}$. Then $\mathbf{p}_1$, $\mathbf{p}_2$ and the sum $\mathbf{p}_1 + \mathbf{p}_2$ are cross-polytopes of m-tuples. In view of Theorem 2.4 and the fact that S_n is transitive on the crosspolytopes, result (i) follows.

Now suppose n is even and $n = 2m$. Consider $\sum_{i=1}^r \mathbf{w}_i$, as above. $\theta_{-m}^{m-1}(\sum_{i=1}^r \mathbf{w}_i) = \sum_i g_n{}^m\{-\mathbf{w}_i\} = \sum_i \{(2m-1, \mathbf{w}_i) + (2m, \mathbf{w}_i)\}$. Thus, $E_n{}^m$ is contained in $\operatorname{Im} \theta_{-m}^{m-1}$. However, $\operatorname{Im} \theta_{-m}^{m-1} = G_n{}^m\{-(m-1)\}$, which by Lemma 2.9 is contained in $E_n^{m\perp}$. Therefore, $E_n{}^m = E_n{}^m \cap E_n^{m\perp}$ in this case.

On the other hand, if $2m < n$, consider a c.p. $\mathbf{p}_1$ on $\{1, 2,..., 2m\}$, one of whose m-tuples is $(1, 2,..., m)$. Act on $\mathbf{p}_1$ by the permutation given by $m+1 \to m+2 \to m+3 \to \cdots \to 2m \to 2m+1 \to m+1$ to obtain a new c.p. $\mathbf{p}_2$, say. Then, the only word $\mathbf{p}_1$ and $\mathbf{p}_2$ have in common is $(1, 2,..., m)$. Therefore, $E_n{}^m$ is not contained in $E_n^{m\perp}$. ∎

3. THE PARITY OF THE BINOMIAL COEFFICIENTS

In this section, a criterion is obtained for the binomial coefficients to be even and several corollaries are noted for future use.

DEFINITIONS. Suppose that a and b are positive integers. We write $\sigma(a)$ for the sum of the digits of a, and $\lambda(a)$ for the length of a, when a is written in binary. That is, $\lambda(a)$ is the integer such that $\lambda(0) = 0$ and for $a > 0$, $2^{\lambda(a)-1} \leqslant a < 2^{\lambda(a)}$. We say that a and b *overlap in binary* if they have a 1 in common. We say *a contains b in binary* if $\lambda(b) < \lambda(a)$ and for each 1 appearing in the binary expansion of b, there is a 1 in the same place in the binary expansion of a.

THEOREM 3.1. *The binomial coefficient* $\binom{n}{r}$ *is even if and only if* r *and* $n - r$ *overlap in binary.*

Proof. Let [ ] here stand for "the integral part of." The highest power of 2 dividing $n!$ is $\sum_{i=1}^{\infty} [n/2^i] = n - \sigma(n)$. Hence, $\binom{n}{r}$ is even if and only if $\sigma(n - r) + \sigma(r) > \sigma(n)$. Then, it is easy to prove that this condition is equivalent to r and $n - r$ overlapping in binary. ∎

An illustration of the usefulness of this criterion is given by

COROLLARY 3.2. $\binom{n-i}{m-i}$ *is even for all* i *between* 0 *and* $m - 1$ *inclusive if and only if* $n \equiv m - 1 \pmod{2^{\lambda(m)}}$.

Proof. Since m has $\lambda(m)$ digits in binary, $m - i$ and $n - m$ overlap in binary for all i between 0 and $m - 1$ if and only if $n - m$ ends in $\lambda(m)$ 1's in binary. This is true if and only if $n - m + 1 \equiv 0 \pmod{2^{\lambda(m)}}$. ∎

This corollary gives a necessary and sufficient condition for $G_n^m\{0\}$ to be contained in E_n^m. Theorem 3.1 also yields the following corollaries.

COROLLARY 3.3. *Suppose* $0 \leqslant j < 2^a$ *and* $j^* = j + 2^a$ *and* 2^{a+1} *divides* $y + 1$ *and* $0 \leqslant x \leqslant \min(y - j^*, 2^a - 1)$. *Then* $\binom{y-x}{j^*}$ *and* $\binom{j+x}{j}$ *are both even or both odd.*

COROLLARY 3.4. *Suppose* $k \geqslant j \geqslant 0$ *and* $\binom{k}{j}$ *is even. Then there is an integer* u *satisfying* $1 \leqslant u \leqslant k$ *and* $\binom{u+j}{j}$ *even and* $\binom{u+k}{k}$ *odd.*

Since E_n^m is such an important module in our calculations, it is nice to known whether any part of the intersection $\bigcap_{i=0}^{m-1} \operatorname{Ker} \theta_i^m$ is superfluous. This question is answered next.

LEMMA 3.5. *Suppose* $0 \leqslant j \leqslant m$ *and* $0 \leqslant k \leqslant m$. *Then* Ker θ_{m-k}^m *contains* Ker θ_{m-j}^m *if and only if* $k \geqslant j$ *and* $\binom{k}{j}$ *is odd.*

Proof. Suppose $k < j$. Let $\mathbf{w}_1 + \cdots + \mathbf{w}_s$ be a c.p. of $(m - j)$-tuples on the points $\{1, 2,..., 2m - 2j\}$. Let $\{a_1,..., a_j\}$, $\{b_1,..., b_j\}$, and $\{1,..., 2m - 2j\}$ be mutually disjoint. Then $\sum_{i=1}^{s} \{(\mathbf{w}_i, a_1,..., a_j) + (\mathbf{w}_i, b_1,..., b_j)\}$ belongs to Ker θ^m_{m-j} but not to Ker θ^m_{m-k}. Thus, we may assume $k \geqslant j$. Assume $\binom{k}{j}$ is odd. Suppose $\mathbf{v}$ is an element of Ker θ^m_{m-j} and that there are r words of $\mathbf{v}$ containing $\{1, 2,..., m - k\}$. Now, $\theta^m_{m-j}(1, 2,..., m)$ has precisely $\binom{k}{j}$ words containing $\{1, 2,..., m - k\}$. Hence, $\theta^m_{m-j}(\mathbf{v}) = \mathbf{0}$ implies that r is even, and thus, $\mathbf{v} \in$ Ker θ^m_{m-k}. Next, suppose $\binom{k}{j}$ is even. Then by Corollary 3.4, there is a u between 1 and k such that $\binom{u+j}{j}$ is even and $\binom{u+k}{k}$ is odd. But this implies that $\mathbf{g}_n{}^m\{-(1, 2,..., n - m - u)\}$ belongs to Ker θ^m_{m-j}, but not to Ker θ^m_{m-k}. This completes the proof. ∎

THEOREM 3.6.

(i) $\bigcap_{i=0}^{m-1}$ Ker $\theta_i{}^m = \bigcap_j$ Ker θ^m_{m-j}, *where the second intersection is over those j which equal 2^a with $0 \leqslant a \leqslant \lambda(m) - 1$.*

(ii) $\bigcap_{i=0}^{m-1}$ Ker $\theta_i{}^m = \bigcap_{i=1}^{m-1}$ Ker $\theta_i{}^m$ *if and only if m is not a power of* 2.

Proof. From Theorem 3.1, there is a j with $0 < j < k$ and $\binom{k}{j}$ odd if and only if k is not a power of 2. The first result then follows from Lemma 3.5. Then, if m is not a power of 2 the intersections in result (ii) are equal. However, if m is a power of 2, then Theorem 3.1 shows that $\binom{2m-1-i}{m-i}$ is even for $0 < i < m$ and odd for $i = 0$. Hence, $\mathbf{g}_n{}^m\{-(1, 2,..., n - 2m + 1)\}$ belongs to Ker $\theta_i{}^m$ for $0 < i < m$ but not for $i = 0$ and the intersections in result (ii) are different. ∎

Note. Provided that $m + 2^{\lambda(m)} - 2 \leqslant n$, the last part of this proof can be adapted to show that no j may be omitted in result (i).

LEMMA 3.7. *Suppose $m \geqslant k \geqslant 0$ and for each i with $0 \leqslant i \leqslant k$, $G_n{}^k\{m\}$ is not contained in* Ker $\theta_i{}^k$. *Then $G_n{}^k\{m\} = V_n{}^k$.*

Proof. Since $G_n{}^k\{m\}$ is not in Ker $\theta_i{}^k$, $\binom{m-i}{k-i}$ is odd for $0 \leqslant i \leqslant k$. Using Lemma 3.5, we get $\bigcap_{i=0}^{k}$ Ker $\theta_i{}^m =$ Ker $\theta_k{}^m$. Now, by Theorem 2.8, $[n] + [n - 1, 1] + \cdots + [n - k, k]$ is the character of both $V_n{}^k$ and $(\bigcap_{i=0}^{k}$ Ker $\theta_i{}^m)^\perp$. The latter is (Ker $\theta_k{}^m)^\perp = G_n{}^m\{k\} \cong G_n{}^k\{m\}$. The result follows. ∎

4. THE MATRICES

A type I matrix is a quarter infinite matrix all of whose entries are 0 or 1. The rule for determining the position of the 1's is best seen by examining the diagram below, in which all the 0's are omitted. It is constructed starting at the bottom right-hand corner, where there is a 1, and working leftwards as follows. Whenever a 1 occurs, there are 1's immediately northwest and southwest from it, except that if this rule gives two 1's in the same place, a 0 is inserted instead.

1										
	1									
1		1								
			1							
1		1		1						
	1				1					
1				1		1				
							1			
				1		1		1		
					1				1	
				1				1		1

Type I.

1										
1	1									
	1	1								
	1	1	1							
1	1		1	1						
1			1	1	1					
			1		1	1				
			1		1	1	1			
			1	1	1		1	1		
			1	1			1	1	1	
			1				1		1	(1)

Type II with the extra 1.
Type III without the extra 1.

1										
2	1									
1	2	1								
	2	2	1							
1	2	1	2	1						
2	1		2	2	1					
1			2	1	2	1				
			2		2	2	1			
			2	1	2	1	2	1		
			2	2	1		2	2	1	
			2	1			2	1	2	1

Type IV.

A type III matrix is constructed by placing a 1 immediately above each 1 in a type I matrix. A type II matrix is obtained by putting a 1 to the bottom right of a type III matrix. A type IV matrix is obtained from a type I matrix by inserting 2's in each column from the top in the following manner: Below the top 1, 2's are placed until the next 1 is met. From this 1 to the third 1 are 0's, below the third 1 to the fourth 1 are 2's, and so on alternately.

A $k \times k$ type I, II or IV matrix is given by the intersection of the bottom k rows and the right-hand k columns of the corresponding infinite matrix. A $k \times (k-1)$ type III matrix is obtained in the same way, counting the first right-hand column as the one with 1's in it.

Definitions. $\alpha(x, m, j)$ is the jth entry in the mth row of an x by x type I matrix. $\beta(x, m, j)$ and $\delta(x, m, j)$ are defined in the same way, replacing "type I" by "type II" or "type IV", respectively. $\gamma(x, m, j)$ is defined in the same way, replacing "x by x type I" by "x by $x-1$ type III."

These matrices have many useful properties, several of which are collected together in the next lemma, the proof of which is straightforward.

Lemma 4.1.

(i) *If* $m-j$ *is odd, then* $\alpha(x, m, j) = 0$.

(ii) *If* $m-j$ *is even, then* $\gamma(x, m, j) = \gamma(x, m+1, j) = \alpha(x, m+1, j+1)$.

(iii) $\beta(x, m, j) = \gamma(x, m, j)$, *unless* $x = m = j$.

(iv) $\alpha(x, m, j) = \alpha(x+1, m+1, j+1)$, *and similarly for* β, γ, *and* δ.

(v) $\beta(x, m, j) + \beta(x, m+1, j) = \delta(x, m+1, j)$ *for* $m \leqslant x-1$.

(vi) $\alpha(x, m+1, 1) = 1$ *if and only if* $2x$ *contains* m *in binary*.

(vii) $\alpha(x, m+1, j) = 0$ *for all* j *between* 1 *and* m *inclusive if and only if* $2x \equiv 2m \pmod{2^{\lambda(m)}}$.

(viii) *For a fixed* m, *suppose that* $\delta(x, m+1, 1) = 2$ *and* j *is least such that* $\delta(x, m+1, j+1) = 1$. *Then*, $\delta(x, j+1, 1) = 2$ *and* $2x \equiv 2j \pmod{2^{\lambda(j)}}$.

(ix) $\delta(x, m, j) = \alpha(x, m, j)$ *if and only if both are* 0 *or* 1.

Several parts of this lemma follow from the fact that the steps described below define all the matrices by induction.

(a) Suppose M_1 is an x by x type I matrix.

(b) Add to each row of M_1, the row below it. This gives an $x+1$ by x type III matrix M_2.

(c) Put a 1 at the bottom right-hand corner of M_2. This gives an $x+1$ by $x+1$ type II matrix M_3.

(d) Add to each row of M_3 the row above it. This gives an $x + 1$ by $x + 1$ type IV matrix M_4.

(e) Omit all the 2's in M_4. This gives an $x + 1$ by $x + 1$ type I matrix, bringing us back to step (a).

At this stage, we illustrate how these steps will fit the matrices into the pattern of things.

Suppose (correctly) that part of the decomposition matrix of S_5 is a 3 by 3 type I matrix:

$$\begin{array}{l|ccc|}
 & \varphi[5] & \varphi[4,1] & \varphi[3,2] \\
{[5]} & 1 & & \\
{[4,1]} & & 1 & \\
{[3,2]} & 1 & & 1
\end{array}\,.$$

Now, $[n - m, m]\downarrow = [n - 1 - m, m] + [n - m, m - 1]$, so looking at restrictions from S_6, we get:

$$\begin{array}{l|ccc|}
 & \varphi[5] & \varphi[4,1] & \varphi[3,2] \\
{[6]\downarrow} & 1 & & \\
{[5,1]\downarrow} & 1 & 1 & \\
{[4,2]\downarrow} & 1 & 1 & 1 \\
{[3,3]\downarrow} & 1 & & 1
\end{array}\,.$$

This 4 by 3 type III matrix turns out to be the same as the matrix of $[6], \ldots, [3, 3]$ by $\varphi[6], \ldots, \varphi[4, 2]$. Therefore, by restricting from S_7, we get

$$\begin{array}{l|ccc|}
 & \varphi[6] & \varphi[5,1] & \varphi[4,2] \\
{[7]\downarrow} & 1 & & \\
{[6,1]\downarrow} & 2 & 1 & \\
{[5,2]\downarrow} & 2 & 2 & 1 \\
{[4,3]\downarrow} & 2 & 1 & 2
\end{array}\,.$$

We prove later that the matrix of $[7], \ldots, [4, 3]$ by $\varphi[7], \ldots, \varphi[4, 3]$ is actually this matrix with the 2's omitted and a 1 put in the bottom right-hand corner; that is, a 4 by 4 type I matrix.

Let M_n be the matrix of $[n], [n - 1, 1], \ldots$ by $\varphi[n], \varphi[n - 1, 1], \ldots$ and M_{n+1} be the corresponding matrix of S_{n+1}. The above examples indicate that if M_n is type I, then M_{n+1} is "at most" type III (we prove that it is type III) and if M_n is type III, then M_{n+1} is "at most" type IV (we prove that it is type I). The first step, that for S_1 the matrix of $[1]$ by $\varphi[1]$ is 1 by 1 type I, is trivial.

5. The Results for Characters of Type $[n-m, m]$

The theorem that enables us to find the constituents of $[n-m, m]$ is

THEOREM 5.1. *If* $n/2 \geqslant m > j \geqslant r \geqslant 0$ *and* $n \equiv m + j - 1 \pmod{2^{\lambda(m-r)}}$, *then* $[n-j, j] \subseteq [n-m, m] + \sum_{j=0}^{r-1} [n-i, i]$.

Proof. By Lemma 2.7, $\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j$ is spanned by elements of the form $\sum_{i=1}^{s} (\mathbf{w}_i, 2r+1,..., j+r)$, where $\sum_{i=1}^{s} \mathbf{w}_i$ is a c.p. of r-tuples on $\{1, 2,..., 2r\}$. Using Lemma 2.5, we find that this element is mapped by θ_{-m}^j to $\sum_{i=1}^{s} \sum_{\mathbf{u}} (\mathbf{u}, \mathbf{w}_i)$, where $\sum_{\mathbf{u}}$ is the sum over $\mathbf{u}$ an $(m-r)$-tuple contained in $\{r+j+1,..., n\}$. Since $n - (r+j) \equiv (m-r) - 1 \pmod{2^{\lambda(m-r)}}$, it follows from Corollary 3.2 that $\theta_{-m}^j(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j)$ is contained in E_n^m. On the other hand, $\theta_{-m}^j(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j)$ is isomorphic to $\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j / \bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j \cap \operatorname{Ker} \theta_{-m}^j$ which, by Theorem 2.1(ii), is isomorphic to

$$G_n^j\{-m\} + \left(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j\right)^{\perp} \Big/ \left(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j\right)^{\perp},$$

which is isomorphic to $G_n^j\{-m\}/(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j)^{\perp} \cap G_n^j\{-m\}$. Now, $G_n^j\{-m\}$ contains $E_n^j = [n-j, j]$ by Lemma 2.6 and $(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j)^{\perp} = [n] + \cdots + [n-r+1, r-1]$ by Theorem 2.8 and the fact that $V_n^j = [n] + \cdots + [n-j, j]$. Combining this information gives the result. ∎

We now come to the statement of the main theorem of this section, which says that the matrix of $[n]$, $[n-1, 1]$, $[n-2, 2]$,... by $\varphi[n]$, $\varphi[n-1, 1]$,... is of type I if n is odd, or type III if n is even.

THEOREM 5.2.

(i) *If* $n = 2x - 1$, *then* $[n-m, m] = \sum_{j=0}^{m} \alpha(x, m+1, j+1)\, \varphi[n-j, j]$.

(ii) *If* $n = 2x$, *then* $[n-m, m] = \sum_{j=0}^{m} \gamma(x+1, m+1, j+1)\, \varphi[n-j, j]$.

The proof of this theorem is in two parts. First the result is shown to be "maximal" in the sense that it describes the greatest number of times $\varphi[n-j, j]$ can be constituent of $[n-m, m]$. Second, Theorem 5.1 is used to show that all the supposed constituents of $[n-m, m]$ really are there.

Assume from now on that Theorem 5.2 is correct for symmetric groups on fewer than n letters.

LEMMA 5.3. *The maximum number of times* $\varphi[n-j, j]$ *can be a constituent of* $[n-m, m]$ *is:*

(i) $\gamma(x+1, m+1, j+1)$ *if* $n = 2x$.

(ii) $\alpha(x, m+1, j+1)$ *if* $n = 2x-1$ *and* $j > 0$.

(iii) $\delta(x, m+1, 1)$ *if* $n = 2x-1$ *and* $j = 0$.

Proof.

$$[n-m, m]\downarrow = [n-m-1, m] + [n-m, m-1]. \tag{1}$$

Case 1. $n = 2x$. The multiplicity of $\varphi[n-j-1, j]$ on the right-hand side of (1) is $\alpha(x, m+1, j+1) + \alpha(x, m, j+1)$ and by Lemma 4.1(i), (ii), and (iv), this equals $\gamma(x+1, m+1, j+1)$. Hence, $\varphi[n-m, m]\downarrow \supseteq \varphi[n-m-1, m]$ for $0 \leqslant m < x$ and result (i) follows.

Case 2. $n = 2x-1$. The multiplicity of $[n-1]$ on the right-hand side of (1) is $\gamma(x, m+1, 1) + \gamma(x, m, 1)$, which by Lemma 4.1(iii) and (v) equal $\delta(x, m+1, 1)$. This proves result (iii).

To prove result (ii), first observe that if $m-j$ is odd, $\varphi[n-j, j]$ belongs to a different block from $[n-m, m]$ and $\alpha(x, m+1, j+1) = 0$. Thus, let us assume $m-j$ is even. Now,

$$[n-m, m]\downarrow\downarrow = [n-m-2, m] + 2[n-m-1, m-1] + [n-m, m-2]. \tag{2}$$

Since the only constituents of $[n-m, m]$ are of the form $\varphi[n-i, i]$ with $m-i$ even, it follows that $\varphi[n-m, m]\downarrow\downarrow \supseteq 2\varphi[n-m-1, m-1]$ for $m > 0$. The multiplicity of $\varphi[n-j-1, j-1]$ on the right-hand side of (2) is $\alpha(x-1, m+1, j) + 2\alpha(x-1, m, j) + \alpha(x-1, m-1, j) = 2\alpha(x, m+1, j+1)$. Therefore, if $j > 0$, the maximum multiplicity of $\varphi[n-j, j]$ in $[n-m, m]$ is $\alpha(x, m+1, j+1)$. This proves result (ii). ∎

If $n = 2x-1$ and $\chi_i = [n-i, i]$, Theorem 5.1 shows that we have the following rule:

$$x > m > j \geqslant r \geqslant 0$$

and

$$2x \equiv m + j \pmod{2^{\lambda(m-r)}} \qquad \text{imply } \chi_j \subseteq \chi_m + \sum_{i=0}^{r-1} \chi_i .$$

Lemma 5.4.

Hypothesis: $x \geqslant 1$ *and* $\chi_0, \chi_1, \ldots, \chi_{x-1}$ *are ascending and the maximum number of times* $\varphi(\chi_j)$ *can be a constituent of* χ_m *is known to be* $\alpha(x, m+1, j+1)$. *We also have the rule described above.*

Conclusion: The multiplicity of χ_0 *in* χ_m *is* $\alpha(x, m+1, 1)$.

Proof. By Lemma 4.1(vi), we must show that if $2x$ contains m in binary, then $\chi_0 \subseteq \chi_m$. This is certainly true if $m = 0$.

Let $I = \{i \mid$ for some m of length i, $2x$ contains m in binary$\}$.

Suppose that $i \in I$ and we have proved that if $\lambda(j) < i$ and $2x$ contains j in binary, then $\chi_0 \subseteq \chi_j$. Then, it is sufficient to prove that $\chi_0 \subseteq \chi_m$ for $m = k - j$, where k is the smallest positive residue of $2x (\bmod 2^i)$. But $k - j \geqslant 2^{i-1} > j \geqslant 0$ and $2x \equiv m + j \pmod{2^{\lambda(m)}}$, so $\chi_0 \subseteq \chi_j \subseteq \chi_m$, as required. ∎

LEMMA 5.5. *Suppose we have the hypothesis of Lemma* 5.4. *Then, the multiplicity of* $\varphi(\chi_j)$ *in* χ_m *is* $\alpha(x, m+1, j+1)$.

Proof. The result is true for $j = 0$ by Lemma 5.4.

For $i > 0$ let

$$\begin{aligned} \chi_i^* &= \chi_i - \chi_0 \qquad \text{if} \qquad \chi_0 \subseteq \chi_i \\ &= \chi_i \qquad\qquad \text{otherwise.} \end{aligned}$$

If $x - 1 > m - 1 > j - 1 \geqslant r \geqslant 0$ and $2(x-1) \equiv (m-1) + (j-1) \pmod{2^{\lambda(m-1-r)}}$, then $\chi_j \subseteq \chi_m + \sum_{i=0}^{r} \chi_i$ and so $\chi_j^* \subseteq \chi_m^* + \sum_{i=1}^{r} \chi_i^*$. Therefore, we get a rule relating to $x - 1$ characters labelled $\chi_1^*, \chi_2^*, \ldots, \chi_{x-1}^*$, and the result follows by induction. ∎

Looking at the proofs of the last two lemmas, it is clear that we have proved Theorem 5.2(i) apart from the multiplicities of $[n]$. The proof will be complete with the following.

LEMMA 5.6. *If* $n = 2x - 1$, *the maximum number of times* $[n]$ *can be a constituent of* $[n - m, m]$ *is* $\alpha(x, m+1, 1)$.

Proof. Suppose that $\delta(x, m+1, 1) = 2$. We must show that $[n]$ is not a constituent of $[n - m, m]$. Let j be the least integer such that $\delta(x, m+j, j+1)$ equals 1. Then $j > 0$, and our results so far show that $[n-m, m] \supseteq \varphi[n-j, j]$. Now, $2x \equiv 2j \pmod{2^{\lambda(j)}}$ and $\delta(x, j+1, 1) = 2$ by Lemma 4.1(viii). Thus, by part (vii) of the same lemma, $\alpha(x, j+1, i) = 0$ for $0 < i \leqslant j$ and hence, $E_n^{\,j} = [n-j, j] = \varphi[n-j, j] + \kappa[n]$, where $\kappa = 0, 1$ or 2. Since Lemma 5.3 shows that $[n-j, j]{\downarrow} \supseteq 2[n-1]$ and $[n-m, m]{\downarrow} \supseteq 2[n-1]$, it will be sufficient to prove that $\kappa = 0$. If $E_n^{\,j}$ contains a submodule of dimension 1, this must be $G_n^{\,j}\{0\}$. But by Corollary 3.2, this would imply $2x - 1 \equiv j - 1 \pmod{2^{\lambda(j)}}$ and hence, $j \equiv 0 \pmod{2^{\lambda(j)}}$, a contradiction. Further, Lemma 2.10(i) shows that $E_n^{\,j}$ does not have a submodule of codimension 1. Therefore, $\kappa = 0$ as required. ∎

The proof of Theorem 5.2(ii) is similar, but easier, because there are no worries about how often $[n]$ is a constituent.

We now list some interesting corollaries of Theorem 5.2.

COROLLARY 5.7.

(i) *For n even,* $[n] - [n-1, 1] + [n-2, 2] - \cdots + (-1)^{n/2}[n/2, n/2]$ *is zero on 2-regular classes.*

(ii) *For n even and $m < n/2$,* $\varphi[n-m, m]\downarrow = \varphi[n-1-m, m]$.

(iii) *For n odd* $= 2x - 1$, $\varphi[x, x-1]\downarrow = 2\varphi[x, x-2]$.

(iv) S_n *has a 2-modular irreducible character of degree* $2^{n/2-1}$ *if n is even, or* $2^{(n-1)/2}$ *if n is odd.*

Note. Using Nakayama's form of Murnaghan's rule involving hooks [2], it is easy to prove that the expression in result (i) is zero on all elements containing a cycle of odd length. Result (iv) (which here follows at once from results (ii) and (iii)), can be deduced fairly easily from result (i).

We have seen that all the constituents of E_n^m besides $\varphi[n-m, m]$ have been deduced from the fact that under certain conditions, $\theta_{-m}^j(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j)$ is contained in E_n^m. But $\operatorname{Im} \theta_{-m}^j = G_n^m\{-j\}$, which is contained in $E_n^{m\perp}$ by Lemma 2.9. Lemma 2.10(ii) shows that $E_n^m \cap E_n^{m\perp}$ is less than E_n^m for $m < n/2$, so we have

THEOREM 5.8. *For* $m < n/2$, $\varphi[n-m, m] = E_n^m / E_n^m \cap E_n^{m\perp}$.

6. THE MODULES W_n^m, H_n^m, AND F_n^m

We now show how the methods used so far can be adapted to deal with the characters of type $[n-m-1, m, 1]$.

For $m > 0$, consider the module W_n^m spanned by elements of the form $(1^*, 2, 3, \ldots, m+1)$; that is by $(m+1)$-tuples containing one special point (called the starred point). Let H_n^m be the submodule consisting of all elements having the property that the link of each starred point belongs to E_{n-1}^m. By construction, H_n^m is isomorphic to

$$E_{n-1}^m \uparrow = [n-m, m] + [n-m-1, m+1] + [n-m-1, m, 1]$$

and

$$H_n^{m\perp} = E_{n-1}^{m\perp} \uparrow.$$

Define the module homomorphisms ψ from W_n^m to V_n^{m+1} and ϵ from V_n^{m+1} to W_n^m by

$$\psi(1^*, 2, \ldots, m+1) = (1, 2, \ldots, m+1)$$

and

$$\epsilon(1, 2, \ldots, m+1) = (1^*, 2, 3, \ldots, m+1) + (1, 2^*, 3, \ldots, m+1) + \cdots + (1, 2, \ldots, (m+1)^*).$$

Then, $\operatorname{Ker}\psi$ is spanned by elements of the form $(1^*, 2, \ldots, m+1) + (1, 2^*, \ldots, m+1)$. Hence, $\operatorname{Im}\epsilon$ is contained in $(\operatorname{Ker}\psi)^\perp$ and if m is odd, $\operatorname{Im}\epsilon$ is contained in $\operatorname{Ker}\psi$. Since ϵ is injective, $\operatorname{Im}\epsilon \cong V_n^{m+1} \cong \operatorname{Im}\psi \cong (\operatorname{Ker}\psi)^\perp$ and so $\operatorname{Im}\epsilon = (\operatorname{Ker}\psi)^\perp$. Also, $\epsilon(\bigcap_{i=1}^{m} \operatorname{Ker}\theta_i^{m+1})$ is contained in $H_n{}^m$ and by Lemma 2.7 and Theorem 2.8, $\psi(H_n{}^m) = \bigcap_{i=0}^{m-1} \operatorname{Ker}\theta_i^{m+1} = [n-m, m] + [n-m-1, m+1]$.

Let $F_n{}^m = H_n{}^m \cap \operatorname{Ker}\psi = [n-m-1, m, 1]$. It is easy to show that $F_n{}^m$ is spanned by elements of the form

$$\sum_{i=1}^{s} \{(1^*, 2, \mathbf{w}_i) + (1^*, 3, \mathbf{w}_i) + (2^*, 1, \mathbf{w}_i) + (2^*, 3, \mathbf{w}_i) + (3^*, 1, \mathbf{w}_i) + (3^*, 2, \mathbf{w}_i)\},$$

where $\sum_{i=1}^{s} \mathbf{w}_i$ is a c.p. of $(m-1)$-tuples, missing 1, 2, and 3.

Henceforth, we shall assume n and m are fixed, with $1 \leqslant m \leqslant n-m-1$ and we write $H = H_n{}^m$ and $F = F_n{}^m$.

Putting the natural bilinear form on $W_n{}^m$, we find that the factors that are labelled the same in the following diagram have the same constituents.

Since we can calculate the constituents of $H \cap H^\perp$ from our knowledge of the constituents of $E_{n-1}^m \cap E_{n-1}^{m\perp}$, we set about determining a and b in diagram V.

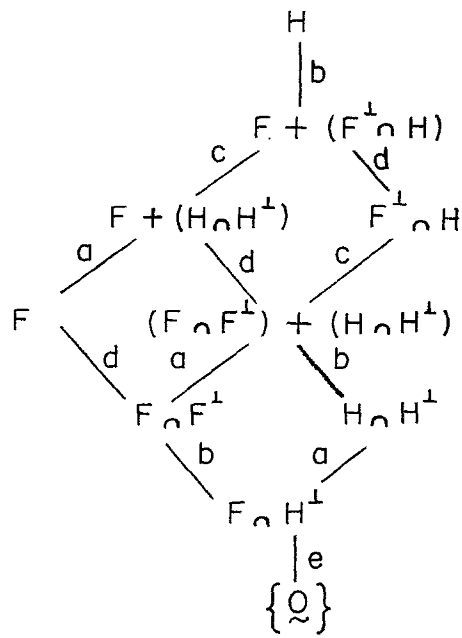


Diagram V

LEMMA 6.1.

(i) *If m is odd, then* $\varphi[n-m-1, m+1] \subseteq b$

(ii) *If* $m+1$ *is a power of* 2, *and* $[n] \subseteq [n-m-1, m+1]$, *then* $\varphi[n-m-1, m+1] + [n] \subseteq b$.

Proof. Let $K = \bigcap_{i=1}^{m} \operatorname{Ker} \theta_i^{m+1}$ and $E = E_n^{m+1} = K \cap \operatorname{Ker} \theta_0^{m+1}$. Since m is odd, $\epsilon(K) \subseteq H \cap \operatorname{Ker} \psi \cap (\operatorname{Ker} \psi)^\perp \subseteq F \cap F^\perp$. It is straightforward to show that if $\mathbf{v}$ is an element of V_n^{m+1} and the link of every point of $\mathbf{v}$ belongs to $E_{n-1}^{m\perp}$, then $\mathbf{v}$ belongs to $E_n^{m+1\perp}$. Thus, $\epsilon^{-1}(F \cap H^\perp) \subseteq \epsilon^{-1}(H^\perp) \subseteq E^\perp$. Therefore, all the constituents of $K/K \cap E^\perp$ are contained in $F \cap F^\perp/F \cap H^\perp = b$. If $m+1$ is not a power of 2, $K = E$ by Theorem 3.6(ii) and result (i) follows from Theorem 5.8.

Thus, assume the hypothesis of result (ii). Now, since K contains E as a submodule of codimension 1, we will be home if we can prove that $K \cap E^\perp = E \cap E^\perp$. If $n = 2x - 1$ then $\alpha(x, m+2, 1) = 1$ and if $n = 2x$, then $\gamma(x+1, m+2, 1) = 1$. Since $m+1$ is a power of 2, Lemma 4.1 can be used to show that there is a j satisfying $m+1 > j \geqslant 0$ and $n \equiv j + m$ (mod $2^{\lambda(m+1)}$) and $n - 2m - 1 \geqslant j$. By Corollary 3.2 and Lemma 2.9, $\mathbf{g}_n^{m+1}\{-(1, 2,..., j)\}, = \mathbf{g}_1$ say, belongs to $E \cap E^\perp$ and $\mathbf{g}_n^{m+1}\{-(1, 2,..., n - 2m - 1)\} = \mathbf{g}_2$ say, belongs to K, but not E. Hence, if there is an element $\mathbf{u}$ in $K \cap E^\perp$, but not $E \cap E^\perp$, then $\mathbf{u} = \mathbf{g}_2 + \mathbf{e}$ for some $\mathbf{e}$ in E. Then $0 = \langle \mathbf{u}, \mathbf{g}_1 \rangle = \langle \mathbf{g}_2, \mathbf{g}_1 \rangle$. This is a contradiction, since every word of $\mathbf{g}_2$ is a word of $\mathbf{g}_1$ and $\mathbf{g}_2$ has an odd number of words. ∎

Note. When $m = 1$, $(1, 2) + (2, 3) + (3, 1)$ belongs to $\operatorname{Ker} \theta_1^2$ and the image of this element under ϵ is $(1^*, 2) + (1^*, 3) + (2^*, 1) + (2^*, 3) + (3^*, 1) + (3^*, 2)$. Since such elements span F_n^1, $F_n^1 = F_n^1 \cap F_n^{1\perp} \cong \operatorname{Ker} \theta_1^2 = [n-2, 2] + [n]$.

The object of the next few lemmas is to prove Theorem 6.2, which determines all the constituents of $F \cap F^\perp$ when n is odd.

THEOREM 6.2. *Assume n is odd and a, b, and c are as described in diagram* V.

(i) *If m is odd and $m+1$ is a power of* 2 *and* $[n] \subseteq [n-m-1, m+1]$, *then* $a = [n-m, m] + [n-m-1, m+1] - \varphi[n-m-1, m+1] - [n]$, $b = \varphi[n-m-1, m+1] + [n]$ *and* $c = 0$.

(ii) *If m is odd and we are not in case* (i), *then* $a = [n-m, m] + [n-m-1, m+1] - \varphi[n-m-1, m+1]$, $b = \varphi[n-m-1, m+1]$ *and* $c = 0$.

(iii) *If m is even, then* $a = [n-m, m] + [n-m-1, m+1] - \varphi[n-m-1, m+1]$, $b = 0$ *and* $c = \varphi[n-m-1, m+1]$.

LEMMA 6.3. *If n is odd, $H \cap H^\perp/F \cap H^\perp$ contains $[n-m, m]$.*

Proof. Let $\sum_{i=1}^{s} \mathbf{w}_i$ be a c.p. of $(m-1)$-tuples on $\{3, 4, \ldots, 2m\}$ and let $\mathbf{u} = \sum_{i=1}^{s} g_n{}^m\{-(1, \mathbf{w}_i)\}$. Then $\mathbf{u}$ belongs to $G_{n-1}^m\{-(m-1)\}$, which by Lemma 2.9 is contained in $E_{n-1}^{m\perp}$. By Lemma 2.5, $\mathbf{u} = \sum_{i=1}^{s} \{(2, \mathbf{w}_i) + (2m+1, \mathbf{w}_i) + (2m+2, \mathbf{w}_i) + \cdots + (n, \mathbf{w}_i)\}$. Since $n - 2m + 1$ is even, $\mathbf{u}$ belongs to $E_{n-1}^m \cap E_{n-1}^{m\perp}$. Let $\mathbf{v}$ be obtained from $\mathbf{u}$ by interchanging 1 and 2. Then $(1^*, \mathbf{u}) + (2^*, \mathbf{v})$ belongs to $H \cap H^\perp$, with the obvious interpretation of $(1^*, \mathbf{u})$ and $(2^*, \mathbf{v})$. But $\psi((1^*, \mathbf{u}) + (2^*, \mathbf{v})) = \sum_{i=1}^{s}\sum_{j=2m+1}^{n} \{(1, j, \mathbf{w}_i) + (2, j, \mathbf{w}_i)\}$. This last element is sent by θ_m^{m+1} to $\sum_{i=1}^{s} \{(1, \mathbf{w}_i) + (2, \mathbf{w}_i)\}$, a c.p. of m-tuples. Thus, $\psi(H \cap H^\perp)$ has a constituent isomorphic to $E_n{}^m$ as required. ∎

We know that $H/F = [n-m, m] + [n-m-1, m+1] = a + b + c$. The lemma just proved shows that $a \supseteq [n-m, m]$, when n is odd. Since $[(n-1)/2, (n+1)/2] = 0$, the proof of Theorem 6.2 is complete in the case when $2m+1 = n$. From now until the proof of Theorem 6.2 is finished, we will assume n is odd and at least $2m+3$.

Next, we set about getting the other parts of $H \cap H^\perp/F \cap H^\perp = a$. In the proof of Theorem 5.8, we saw that all the constituents of $E_n^{m+1} \cap E_n^{m+1\perp}$ turn up as images under $\theta_{-(m+1)}^j$ of $\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i{}^j$ for some $0 \leqslant r \leqslant j < m+1 \leqslant n/2$. For n odd, $\varphi[n-m, m]$ is not a constituent of $[n-m-1, m+1]$, because it is in the wrong block. Thus, we may strengthen the inequality $j < m+1$ to $j < m$. If $n \equiv j + m \pmod{2^{\lambda(m+1)}}$ and $n \equiv j - 1 \pmod{2^{\lambda(j)}}$, then $[n] \subseteq [n-j, j] \subseteq [n-m-1, m+1]$ by Theorem 5.1. Further, if $m+1$ is a power of 2, b will then contain $[n]$ by Lemma 6.1. Since $H/F = [n-m, m] + [n-m-1, m+1]$, H/F contains each constituent (including $[n]$) with multiplicity 1. Hence, results (i) and (ii) of Theorem 6.2 will be proved when we have shown

LEMMA 6.4. *Suppose n is odd, $(n-3)/2 \geqslant m > j \geqslant r \geqslant 0$ and $n \equiv m + j \pmod{2^{\lambda(m+1-r)}}$*

(i) *If $r = 0$ and $n \equiv j - 1 \pmod{2^{\lambda(j)}}$ and $m+1$ is a power of 2, then $\psi(H \cap H^\perp)$ has a submodule isomorphic to a module of codimension at most 1 in $\theta_{-(m+1)}^j(V_n{}^j)$.*

(ii) *In all other cases, $\psi(H \cap H^\perp)$ has a submodule isomorphic to $\theta_{-(m+1)}^j(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i{}^j)$.*

Proof. *Case* 1, $r > 0$. Let $\sum_{i=1}^{s} \mathbf{w}_i$ be a c.p. of $(r-1)$-tuples on $\{3, 4, \ldots, 2r\}$ and suppose k satisfies $r + j + 1 \leqslant k \leqslant n$. Let $\mathbf{v}_k = \sum_{i=1}^{s} \mathbf{g}_n{}^m\{-(k, 1, \mathbf{w}_i, 2r+1, 2r+2, \ldots, r+j)\}$. By Lemma 2.5 $\mathbf{v}_k = \sum_{i=1}^{s}\sum_{\mathbf{u}} (\mathbf{u}, \mathbf{w}_i)$, where $\sum \mathbf{u} = \mathbf{g}_n^{m+1-r}\{-(k, 1, 3, 4, \ldots, r+j)\}$. Since $m > j$

and $n-j-r \equiv m-r \pmod{2^{\lambda(m+1-r)}}$, $\mathbf{v}_k$ belongs to $E_{n-1}^m \cap E_{n-1}^{m\perp}$, where the $n-1$ points are $\{2, 3,\ldots, n\}$. Let $\mathbf{v}_k'$ be obtained from $\mathbf{v}_k$ by interchanging 1 and 2. Add 1^* to each word of $\mathbf{v}_k$ and 2^* to each word of $\mathbf{v}_k'$ and act on the sum of the resulting elements by ψ. The element $\mathbf{u}_k$ of $\psi(H \cap H^\perp)$ is obtained, where $\mathbf{u}_k = \sum_{i=1}^s \sum_{\mathbf{t}} \{(1, \mathbf{w}_i, \mathbf{t}) + (2, \mathbf{w}_i, \mathbf{t})\}$ and $\sum_{\mathbf{t}} = \mathbf{g}_n^{m+1-r}\{-(k, 1, 2,\ldots, r+j)\}$. Now, $\sum_{i=1}^s \{(1, \mathbf{w}_i) + (2, \mathbf{w}_i)\}$ is a c.p. of r-tuples on $\{1, 2,\ldots, 2r\}$ and equals $\sum_{i=1}^{2s} \mathbf{w}_i'$, say. Since $n-(r+j)-(m+1-r)$ is odd, if $\sum_{\mathbf{t}'} = \mathbf{g}_n^{m+1-r}\{-(1, 2,\ldots, r+j)\}$, then $\sum_{k=r+j+1}^n \mathbf{u}_k = \sum_{i=1}^{2s} \sum_{\mathbf{t}'} (\mathbf{w}_i', \mathbf{t}')$. By Lemma 2.5, this equals $\sum_{i=1}^{2s} \mathbf{g}_n^{m+1}\{-(\mathbf{w}_i', 2r+1, 2r+2,\ldots, r+j)\}$. But $\theta_{-(m+1)}^j(\bigcap_{i=0}^{r-1} \operatorname{Ker} \theta_i^j)$ is spanned by such objects, by Lemma 2.7 and the result is proved in this case.

Case 2, $r=0$ *and* $n \not\equiv j-1 \pmod{2^{\lambda(j)}}$. Then $j>0$ and $n \equiv m+j \pmod{2^{\lambda(m+1-1)}}$, so by Case 1, $\psi(H \cap H^\perp)$ contains $\theta_{-(m+1)}^j(\operatorname{Ker} \theta_0^j) \cong \operatorname{Ker} \theta_0^j / \operatorname{Ker} \theta_0^j \cap \operatorname{Ker} \theta_{-(m+1)}^j$. The conditions and Lemma 2.6 imply that $G_n^j\{0\} \not\subseteq E_n^j = G_n^j\{-(m+1)\}$. Hence, by Theorem 2.1, $\operatorname{Ker} \theta_0^j$ does not contain $\operatorname{Ker} \theta_{-(m+1)}^j$. Since $\operatorname{Ker} \theta_0^j$ is of codimension 1 in V_n^j, it follows that $\theta_{-(m+1)}^j(\operatorname{Ker} \theta_0^j) \cong \theta_{-(m+1)}^j(V_n^j)$, as required.

This proof also shows that when $r=0$, $\psi(H \cap H^\perp)$ has a submodule isomorphic to a module of codimension at most 1 in $\theta_{-(m+1)}^j(V_n^j)$. Thus, we are left to prove the result in the following.

Case 3, $r=0$ *and* $n \equiv j-1 \pmod{2^{\lambda(j)}}$ *and* $m+1$ *not a power of* 2. Let $j^* = j + 2^{\lambda(m+1)-1}$ and define k by $k-j = m+1-2^{\lambda(m+1)-1}$. Since $2^{\lambda(j)} \leqslant 2^{\lambda(m+1)}$, then $2^{\lambda(j)}$ divides both $n-m-j$ and $n-j+1$ and hence, also $m+1$. Because $m+1$ is not a power of 2, it follows that $j^* < m+1 \leqslant 2^{\lambda(m+1)} - 2^{\lambda(j)} < 2^{\lambda(m+1)} - j$. If $j^* = m$, then $m+j$ is even, contradicting n odd and congruent to $m+j \pmod 2$. Therefore, $m > j^* \geqslant k-j+1 > 0$. By Case 1, $\psi(H \cap H^\perp)$ contains $\theta_{-(m+1)}^{j^*}(\bigcap_{i=0}^{k-j} \operatorname{Ker} \theta_i^{j^*})$.

Now, in binary, $n-m-j$ ends in at least $\lambda(m+1)$ zeros. Thus, $n-m-1-2j$ has 1's in the xth place for $\lambda(j)+1 \leqslant x \leqslant \lambda(m+1)$. But j^* has $\lambda(m+1)$ digits and $j^*-(k-j)$ has at least $\lambda(j)+1$ digits. Therefore, for $0 \leqslant i \leqslant k-j$, $n-m-1-2j$ $(=n-k-j^*)$ and j^*-i overlap in binary. Theorem 3.1 implies $\binom{n-k-i}{j^*-i}$ are all even. Therefore, $G_n^{j^*}\{-k\}$ is contained in $\bigcap_{i=0}^{k-j} \operatorname{Ker} \theta_i^{j^*}$. Thus, if $\mathbf{v}$ is a fixed k-tuple, $\sum \{\mathbf{g}_n^{m+1}\{-\mathbf{w}\} \mid \mathbf{w} \in \mathbf{g}_n^{j^*}\{-\mathbf{v}\}\}$, $= \mathbf{u}_1$ say, belongs to $\psi(H \cap H^\perp)$. Next, we show that $\mathbf{u}_1 = \mathbf{u}_2$, where $\mathbf{u}_2 = \sum\{\mathbf{g}_n^{m+1}\{-\mathbf{w}\} \mid \mathbf{w} \in \mathbf{g}_n^j\{\mathbf{v}\}\}$.

Suppose that $\mathbf{s}$ is an $(m+1)$-tuple such that $|\mathbf{s} \cap \mathbf{v}| = k-j-x$ with $-j \leqslant x \leqslant k-j$ (so $|\mathbf{s} \cup \mathbf{v}| = m+1+j+x$). Then $\mathbf{s} \in \mathbf{u}_1$ if and only if $\binom{n-m-1-j-x}{j^*}$ is odd. But for $-j \leqslant x < 0$, $n-m-1-x-j^*$ and j^* overlap in binary. Thus, $\mathbf{s} \in \mathbf{u}_1$ if and only if $\binom{n-m-1-j-x}{j^*}$ is odd and $0 \leqslant x \leqslant k-j$. Now, $n \geqslant m+j+2^{\lambda(m+1)} > 2m+2j+1$. Hence, $k-j \leqslant \min(n-m-j-j^*-1, 2^{\lambda(m+1)-1}-1)$. Also, $0 \leqslant j < 2^{\lambda(m+1)-1}$ and $2^{\lambda(m+1)}$

divides $n - m - j$. By Corollary 3.3, $\mathbf{s} \in \mathbf{u}_1$ if and only if $\binom{j+x}{j}$ is odd and $0 \leqslant x \leqslant k - j$. That is, if and only if $\mathbf{s} \in \mathbf{u}_2$.

We have now proved that $\mathbf{u}_2 \in \psi(H \cap H^\perp)$. Since $2^{\lambda(j)}$ divides $k - j$, it follows that for $0 \leqslant i \leqslant j - 1$ $\binom{k-i}{j-i}$ is odd. By Lemma 3.7, $G_n{}^j\{k\} = V_n{}^j$. By letting $\mathbf{v}$ vary, we deduce that $\psi(H \cap H^\perp)$ contains $G_n^{m+1}\{-j\}$, which equals $\theta^j_{-(m+1)}(V_n{}^j)$, as required. ∎

It remains to prove result (iii) of Theorem 6.2. Suppose m is even. We still have $a \supseteq [n - m, m] + [n - m - 1, m + 1] - \varphi[n - m - 1, m + 1]$ by Lemmas 6.3 and 6.4 and we have to decide whether a, b, or c contains the final constituent $\varphi[n - m - 1, m + 1]$. Since $\varphi[n - m - 1, m + 1]$ is not in the same block as $[n - m - 1, m, 1] = F$, then $b = 0$. Now, $\varphi[n - m - 1, m + 1]\!\downarrow$ contains $\varphi[n - m - 2, m + 1]$. The constituents of $E^m_{n-1} \cap E^{m\perp}_{n-1}$ are of the form $\varphi[n - 1 - i, i]$ with $i < m$, and for $i < m$ $[n - i - 1, i]\!\uparrow\downarrow$ does not have $\varphi[n - m - 2, m + 1]$ as a constituent. Since $H \cap H^\perp = E^m_{n-1} \cap E^{m\perp}_{n-1}\!\uparrow$, it follows that a does not contain $\varphi[n - m - 1, m + 1]$ and so $c = \varphi[n - m - 1, m + 1]$. This completes the proof of Theorem 6.2.

7. The Results for Characters of Type $[n - m - 1, m, 1]$

We start by defining numbers $\kappa(x, m)$, which will count the multiplicity of $[n]$.

Definitions.

(i) $\mu(x, m) = 2\alpha(x, m + 1, 1) + 2\alpha(x, m + 2, 1)$ if $1 \leqslant m \leqslant x - 1$ and $m + 1$ is not a power of 2.
$= 0$ otherwise.

(ii) $\nu(x, m) = 2\alpha(x, m + 2, 1)$ if $1 \leqslant m \leqslant x - 1$ and $m + 1$ is a power of 2.
$= 1$ if $m = 0$,
$= 0$ otherwise.

(iii) $\kappa(x, m) = \sum_{i=1}^{m} \gamma(x + 1, m + 1, i)\, \nu(x + 1, i - 1) - \mu(x + 1, m)$.

Recalling that for $x \geqslant y$, $\varphi[y, x, 1] = \varphi[n - 2, 1, 1] = 0$, by definition, we state

Theorem 7.1. *For $n = 2x + 1$, the constituents of $[n - m - 1, m, 1]$ for $1 \leqslant m \leqslant x$ are as follows:*

(i) *For m even*, $[n-m-1, m, 1] = \sum_{j=2}^{m} \gamma(x, m, j)\, \varphi[n-j-1, j, 1] + [n-m, m] + \kappa(x, m)\, [n]$.

(ii) *For m odd*, $[n-m-1, m, 1] = \sum_{j=2}^{m} \gamma(x, m, j)\, \varphi[n-j-1, j, 1] + [n-m-1, m+1] + \kappa(x, m)\, [n]$.

For $n = 2x$, and $1 \leqslant m \leqslant x-1$, we have

(iii) *For m even*, $[n-m-1, m, 1] = \sum_{j=2}^{m} \alpha(x, m+1, j+1)\, \varphi[n-j-1, j, 1]$.

(iv) *For m odd*, $[n-m-1, m, 1] = \sum_{j=2}^{m} \alpha(x, m+1, j+1)\, \varphi[n-j-1, j, 1] + [n-m-1, m+1] + \sum_{i=1}^{m-1} \alpha(x, m, i+1)\, \varphi[n-i, i] + \kappa(x, m)\, [n]$.

It is possible to restate this theorem in terms involving only type I–IV matrices, temporarily ignoring the multiplicity of $[n]$. For instance, the matrix of $[n-2, 1, 1], [n-4, 3, 1], [n-6, 5, 1], \ldots$ by $\varphi[n-2, 2], \varphi[n-4, 4], \ldots$ is k by k type II if $n = 4k+1$, k by k type I if $n = 4k+3$, k by $k-1$ type III if $n = 4k$, or k by k type IV if $n = 4k+2$. However, by choosing x and m carefully, we can make $\kappa(x, m)$ as large as we please.

Proof of Theorem 7.1. It is tedious, but straightforward to check by methods similar to those in Lemma 5.3, that the multiplicities of all the modular irreducible characters, except perhaps $[n]$, in $[n-m-1, m, 1]$ can be at most the numbers claimed in the theorem. Therefore, we assume that the given multiplicities, except that of $[n]$, are maximal.

Case 1, $n = 2x+1$. We prove by induction, that for $1 \leqslant m \leqslant x$, $F/F \cap F^{\perp} = \varphi[n-m-1, m, 1]$. This is true for $m = 1$ by the note to Lemma 6.1, so assume it is true for $j < m$. Since $\varphi[n-j-1, j]\uparrow = H_n^{\,j}/H_n^{\,j} \cap H_n^{j\perp}$ for $1 \leqslant j < m$, we get from Theorem 6.2 that

$$\varphi[n-j-1, j]\uparrow = \varphi[n-j-1, j, 1] + 2\varphi[n-j-1, j+1] + \nu(x+1, j)\, [n] \quad \text{for } j \text{ odd.}$$

and

$$\varphi[n-j-1, j]\uparrow = \varphi[n-j-1, j, 1] + \varphi[n-j-1, j+1] + \nu(x+1, j)\, [n] \quad \text{for } j \text{ even.}$$

This is also true if $j = 0$.

Now assume m is odd. Then, by Theorems 5.2 and 5.8, $H \cap H^{\perp} = E_{n-1}^{m} \cap E_{n-1}^{m\perp}\uparrow = \sum_{j=0}^{m-1} \gamma(x+1, m+1, j+1)\, \varphi[n-j, j]\uparrow$. This equals $\sum_{j=2}^{m-1} \gamma(x+1, m+1, j+1)\, \varphi[n-j-1, j, 1] + \sum_{i=1}^{(m+1)/2} \gamma(x+1, m+1, 2i-1)\, \varphi[n-2i+1, 2i-1] + 2\sum_{i=1}^{(m-1)/2} \gamma(x+1, m+1, 2i)\, \varphi[n-2i, 2i] + \sum_{j=0}^{m-1} \gamma(x+1, m+1, j+1)\, \nu(x+1, j)\, [n]$.

Now, $\gamma(x+1, m+1, 2i-1) = \alpha(x+1, m+1, 2i)$ and $\gamma(x+1, m+1, 2i) = \alpha(x+1, m+2, 2i+1)$. Using Theorem 5.2, we get

$H \cap H^\perp = \sum_{j=2}^{m-1} \gamma(x, m, j)\, \varphi[n-j-1, j, 1] + [n-m, m] + 2[n-m-1, m+1] - 2\varphi[n-m-1, m+1] + \{\sum_{i=1}^{m} \gamma(x+1, m+1, i)\, \nu(x+1, i-1) - 2\alpha(x+1, m+2, 1)\}\, [n]$. By Theorem 6.2, $F \cap F^\perp = H \cap H^\perp - [n-m, m] - [n-m-1, m+1] + 2\varphi[n-m-1, m+1] + r[n]$, where r equals 2 if $m+1$ is a power of 2 and $\alpha(x+1, m+2, 1) = 1$ and r equals 0 otherwise. Therefore, $2\alpha(x+1, m+2, 1) - r = \mu(x+1, m)$. Hence $F \cap F^\perp = \sum_{j=2}^{m-1} \gamma(x, m, j)\, \varphi[n-j-1, j, 1] + [n-m-1, m+1] + \kappa(x, m)\, [n]$. Thus, all the possible constituents of $[n-m-1, m, 1]$ lie in $F \cap F^\perp$ except $\varphi[n-m-1, m, 1]$ and maybe some further multiple of $[n]$. But it is easy to show, as in Lemma 2.10, that F does not have a submodule of codimension 1. Therefore, $F/F \cap F^\perp = \varphi[n-m-1, m, 1]$ and the result follows.

In the case m is even, the result follows in exactly the same way.

Case 2, $n = 2x$. We may assume results (i) and (ii) are true for S_{n+1}, by Case 1.Result (iv) is true for $m = 1$ by the note to Lemma 6.1. We assume that for $1 \leqslant j < m$

(a) The constituents of $[n-j-1, j, 1]$ are given by results (iii) and (iv).

(b) For j even $\geqslant 2$, $\varphi[n-j, j, 1]\!\downarrow\, = \varphi[n-j-1, j, 1] + \varphi[n-j, j]$.

(c) For j odd $\geqslant 1$, $\varphi[n-j, j, 1]\!\downarrow\, = \varphi[n-j-1, j, 1]$.

For the moment, assume m is even. An alternative expression for $[n-m, m-1, 1]$ is obtained first. We have $\mu(x+1, m) = 2\alpha(x+1, m+1, 1) = \nu(x+1, m-1) + \mu(x+1, m-1)$. Since $\nu(x+1, i)$ is 0 for i even > 0, $\kappa(x, m) - \gamma(x+1, m+1, 1) = \sum_{i=1}^{m/2} \gamma(x+1, m+1, 2i)\, \nu(x+1, 2i-1) - \mu(x+1, m) = \nu(x+1, m-1) + \sum_{i=1}^{(m-2)/2} \gamma(x+1, m, 2i)\, \nu(x+1, 2i-1) - \mu(x+1, m) = \kappa(x, m-1) - \gamma(x+1, m, 1)$. Hence, for m even between 2 and $x-1$,

$$[n-m, m] + \sum_{i=1}^{m-2} \alpha(x, m-1, i+1)\, \varphi[n-i, i] + \kappa(x, m-1)\, [n]$$

$$= [n-m+1, m-1] + \sum_{i=2}^{m} \alpha(x, m+1, i+1)\, \varphi[n-i, i] + \kappa(x, m)\, [n].$$

Still assuming m even, the result for S_{n+1} gives $[n-m, m, 1] = \sum_{j=2}^{m} \gamma(x, m, j)\, \varphi[n-j, j, 1] + [n-m+1, m] + \kappa(x, m)\, [n+1]$, so $[n-m, m, 1]\!\downarrow\, = [n-m-1, m, 1] + [n-m, m-1, 1] + [n-m, m] = [n-m, m] + [n-m+1, m-1] + \varphi[n-m, m, 1]\!\downarrow + \kappa(x, m)\, [n] + \sum_{i=1}^{(m-2)/2} \gamma(x, m, 2i)\, \varphi[n-2i, 2i] + \sum_{i=1}^{(m-2)/2} \{\gamma(x, m, 2i)\, \varphi[n-2i-1, 2i, 1] +$

$\gamma(x, m, 2i + 1)\,\varphi[n - 2i - 2, 2i + 1, 1]\}$, using the induction hypotheses (b) and (c). Now, by Lemma 4.1(i) and (ii),

$$\sum_{i=1}^{(m-2)/2} \gamma(x, m, 2i + 1)\,\varphi[n - 2i - 2, 2i + 1, 1]$$

$$= \sum_{j=2}^{m-1} \alpha(x, m, j + 1)\,\varphi[n - j - 1, j, 1],$$

which, by the induction hypothesis (a) and the alternative expression for $[n - m, m - 1, 1]$ derived above, equals

$$[n - m, m - 1, 1] - [n - m + 1, m - 1] - \kappa(x, m)\,[n]$$
$$- \sum_{i=2}^{m} \alpha(x, m + 1, i + 1)\,\varphi[n - i, i].$$

Again, using Lemma 4.1, we deduce that

$$\varphi[n - m, m, 1]\!\downarrow\; = [n - m - 1, m, 1] + \varphi[n - m, m]$$
$$- \sum_{j=2}^{m-1} \alpha(x, m + 1, j + 1)\,\varphi[n - j - 1, j, 1].$$

Therefore, all the constituents that we claim to be inside $[n - m - 1, m, 1]$ are in fact constituents and so the decomposition of $[n - m - 1, m, 1]$ is correct. (Note that $[n]$ is not a constituent as it is in the wrong block.) Also, $\varphi[n - m, m, 1]\!\downarrow\; = \varphi[n - m - 1, m, 1] + \varphi[n - m, m]$, which gives the next inductive step (b).

Next, assume m is odd. We restrict $[n - m, m, 1]$ from S_{n+1} as above, and this time obtain

$$\varphi[n - m, m, 1]\!\downarrow\; = [n - m - 1, m, 1] - [n - m - 1, m + 1] - \kappa(x, m)\,[n]$$
$$- \sum_{j=2}^{m-1} \alpha(x, m + 1, j + 1)\,\varphi[n - j - 1, j, 1]$$
$$- \sum_{j=1}^{m-1} \alpha(x, m, j + 1)\,\varphi[n - j, j].$$

Thus, again we get all the constituents of $[n - m - 1, m, 1]$ that we claimed were there. By the result for the maximality of these constituents, $\varphi[n - m, m, 1]\!\downarrow\; = \varphi[n - m - 1, m, 1] +$ (some multiple of $[n]$). But this multiple of $[n]$ must be zero by Theorem C of [1]. This concludes the proof of Theorem 7.1. ∎

8. Applications

In this section, we show how to apply the results of Theorems 5.2 and 7.1 to check Robinson's results [4] for the 2-modular irreducible characters of S_n with $n = 8$ and 9. When $n \leqslant 7$, the theorems are even easier to apply. Finally, all the 2-modular irreducibles of S_{10} are calculated.

Case 1, $n = 8$. From Theorem 5.2, we have at once that part of the decomposition matrix is of type III:

$$\left[\begin{array}{lcccc} & \varphi[8] & \varphi[7,1] & \varphi[6,2] & \varphi[5,3] \\ {[8]} & 1 & & & \\ {[7,1]} & 1 & 1 & & \\ {[6,2]} & & 1 & 1 & \\ {[5,3]} & & 1 & 1 & 1 \\ {[4,4]} & & 1 & & 1 \end{array}\right].$$

A quick calculation shows that $\kappa(4, 3) = 2$. By Theorem 7.1,

$$[5, 2, 1] = \alpha(4, 3, 3)\,\varphi[5, 2, 1] = \varphi[5, 2, 1].$$

$$\begin{aligned}[4, 3, 1] &= \alpha(4, 4, 4)\,\varphi[4, 3, 1] + [4, 4] + \alpha(4, 3, 2)\,\varphi[7, 1] + \alpha(4, 3, 3)\,\varphi[6, 2] \\ &\quad + \kappa(4, 3)\,[8] \\ &= \varphi[4, 3, 1] + \varphi[7, 1] + \varphi[6, 2] + \varphi[5, 3] + 2[8].\end{aligned}$$

These equations give the last two modular irreducibles $\varphi[5, 2, 1]$ and $\varphi[4, 3, 1]$.

Case 2, $n = 9$. Theorem 5.2 shows that the part of the decomposition matrix corresponding to the characters of the form $[n - m, m]$ is type I:

$$\left[\begin{array}{lccccc} & \varphi[9] & \varphi[8,1] & \varphi[7,2] & \varphi[6,3] & \varphi[5,4] \\ {[9]} & 1 & & & & \\ {[8,1]} & & 1 & & & \\ {[7,2]} & 1 & & 1 & & \\ {[6,3]} & & & & 1 & \\ {[5,4]} & & & 1 & & 1 \end{array}\right].$$

Now, $\kappa(4, 2) = 0$ and $\kappa(4, 3) = 2$, so

$$\begin{aligned}[6, 2, 1] &= \gamma(4, 2, 2)\,\varphi[6, 2, 1] + [7, 2] + \kappa(4, 2)\,[9] = \varphi[6, 2, 1] + \varphi[7, 2] \\ &\quad + [9].\end{aligned}$$

$$\begin{aligned}[5, 3, 1] &= \gamma(4, 3, 2)\,\varphi[6, 2, 1] + \gamma(4, 3, 3)\,\varphi[5, 3, 1] + [5, 4] + \kappa(4, 3)\,[9]. \\ &= \varphi[6, 2, 1] + \varphi[5, 3, 1] + \varphi[7, 2] + \varphi[5, 4] + 2[9].\end{aligned}$$

Thus, $\varphi[6, 2, 1]$ and $\varphi[5, 3, 1]$ are given by these equations and there is one modular irreducible left to be found, namely, $\varphi[4, 3, 2]$.

Since $[4, 3, 2]$ has 2-core $[2, 1]$, the only possible constituents of $[4, 3, 2]$ are $[8, 1]$ and $[6, 3]$ (both of which are irreducible).

Inducing up $\varphi[4, 3, 1] = [4, 3, 1] - [5, 3] - 2[8]$ from S_8 and ignoring characters in the principal block, we find that there is a modular character $[4, 3, 2] + [4, 3, 1^2] - [6, 3] - 2[8, 1]$. Referring to the character table of S_9, we find that $[4, 3, 1^2] = [4, 3, 2] + [6, 3]$ on 2-regular classes. Hence, there is a modular character $[4, 3, 2] - [8, 1]$. Now $[4, 3, 2]{\downarrow} = 6[8] + \varphi[7, 1] + 2\varphi[6, 2] + \varphi[5, 3] + 3\varphi[4, 3, 1]$. But $[8, 1]{\downarrow} = 2[8] + \varphi[7, 1]$ and $[6, 3]{\downarrow} = 2\varphi[7, 1] + 2\varphi[6, 2] + \varphi[5, 3]$. Looking at the multiplicity of $\varphi[7, 1]$, we deduce that $[4, 3, 2] - [8, 1]$ is the last irreducible $\varphi[4, 3, 2]$ we are seeking. Thus, we have checked Robinson's results for $n = 8$ and 9.

Case 3, $n = 10$. This time we get

$$\begin{array}{c|ccccc}
 & \varphi[10] & \varphi[9,1] & \varphi[8,2] & \varphi[7,3] & \varphi[6,4] \\
\hline
[10] & 1 & & & & \\
[9,1] & 1 & 1 & & & \\
[8,2] & 1 & 1 & 1 & & \\
[7,3] & 1 & & 1 & 1 & \\
[6,4] & & & 1 & 1 & 1 \\
[5,5] & & & 1 & & 1
\end{array}.$$

Also

$$[7, 2, 1] = \varphi[7, 2, 1],$$
$$[5, 4, 1] = \varphi[7, 2, 1] + \varphi[5, 4, 1],$$

and

$$[6, 3, 1] = \varphi[6, 3, 1] + 2\varphi[8, 2] + \varphi[7, 3] + \varphi[6, 4] + [10].$$

Since it is in a block by itself, $[4, 3, 2, 1]$ is irreducible.

The only irreducible left to be calculated is $\varphi[5, 3, 2]$. For this, first observe from the character table that $[5, 3, 2] = [4, 3^2] + \varphi[6, 3, 1] + \varphi[8, 2] + \varphi[6, 4]$ on 2-regular classes. Also, $\varphi[5, 3, 1] = [3^3] - 2[9]$. Noting that $[4, 3^2] = [3^3, 1]$ on 2-regular classes, we get that

$$\varphi[5, 3, 1]{\uparrow} = 2[4, 3^2] - 2[10] - 2[9, 1] = 2[4, 3^2] - 4[10] - 2\varphi[9, 1].$$

Hence, there is a modular character $[5, 3, 2] - \varphi[6, 3, 1] - \varphi[8, 2] - \varphi[6, 4] - 2[10] - \varphi[9, 1], = \chi$, say. The only possible constituents of χ besides $\varphi[5, 3, 2]$, are of the form $\varphi[10 - m, m]$, or $\varphi[6, 3, 1]$. But $\varphi[10 - m, m]{\downarrow} = \varphi[9 - m, m]$, $\varphi[6, 3, 1]{\downarrow} = 2[9] + 2\varphi[6, 2, 1] + \varphi[5, 3, 1]$

and $\chi\downarrow = \varphi[5, 3, 1] + \varphi[4, 3, 2]$. Hence, χ is irreducible and must be $\varphi[5, 3, 2]$.

If the full decomposition matrix is needed, it can be worked out by finding how each ordinary irreducible character restricted to 2-regular classes can be written in terms of the modular irreducibles. Alternatively, Robinson's methods give a set of columns spanning the same space as the columns of the decomposition matrix, and since the right number of linearly independent rows has been found, the full decomposition matrix can rapidly be calculated.

Acknowledgment

The author, who proved Theorem 1.1 for G soluble or a symmetric group, is indebted to Professor G. Glauberman for the proof given with that theorem for an arbitrary finite group.

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